Fourier Analysis 03-12

Review.

Def. A sequence of numbers $(x_n)_{n=1}^{\infty}$ in $[0,1)$ is said to be equidistributed in $[0,1)$ if the following holds: $\forall\ (a,b) \subset [0,1)$,

$$① \quad \lim_{N\to\infty} \frac{1}{N} \#\left\{1 \leq n \leq N : \quad x_n \in (a,b)\right\} = b-a.$$

Thm 1 (Weyl) Let γ be an irrational number.

Then the sequence $(\{n\gamma\})_{n=1}^{\infty}$ is equidistributed in $[0,1)$. Here $\{x\}$ denotes the fractional part of x.

More general, Weyl proved the following result:

Thm 2 (Weyl's criterion). Let $(x_n)_{n=1}^{\infty} \subset [0,1)$.

Then (x_n) is equidistributed in $[0,1)$ if and only if

$$(1) \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} = 0 \text{ for all } k \in \mathbb{Z}\setminus\{0\}.$$

Pf. We first prove the necessity.

We assume that (x_n) is equidistributed in $[0,1)$.

By definition, we have

$$(2) \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(x_n) = \int_0^1 \chi_{(a,b)}(x)\,dx, \quad \forall (a,b) \subset [0,1).$$

It follows that for any step function f of the form

$$f(x) = \sum_{i=1}^{\ell} c_i \chi_{(a_i,b_i)}(x),$$

we have

$$(3) \quad \lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx.$$

We claim that (3) also holds for any $f \in \mathcal{R}[0,1)$.

To see it, let $f \in \mathcal{R}[0,1)$. Let $\varepsilon > 0$. Then $\exists$ two step functions, say g^+, g^- such that

$$g^- \le f \le g^+$$

and

$$\int_0^1 g^+ - f\,dx < \varepsilon, \quad \int_0^1 f - g_-\,dx < \varepsilon.$$

Now

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} f(x_n) \leqslant \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} g^+(x_n)$$

$$= \int_0^1 g^+ dx < \int_0^1 f\, dx + \varepsilon .$$

Similarly

$$\underline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} f(x_n) \geqslant \underline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} g^-(x_n)$$

$$= \int_0^1 g^- dx > \int_0^1 f\, dx - \varepsilon .$$

Since ε is arbitrarily taken, we see that ③ holds for any $f \in \mathcal{R}[0,1)$. In particular, taking $f = e^{2\pi i k x}$ in ③ we obtain

$$(1) \quad \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} \longrightarrow \int_0^1 e^{2\pi i k x} dx = 0 \quad \text{if } k \in \mathbb{Z} \setminus \{0\} .$$

- Next we prove the sufficiency.

Suppose (1) holds. Then for any trigonometric polynomial

$$f = \sum_{|n| \leq N} c_n e^{2\pi i n x} ,$$

We have

$$(3) \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx$$

Then by Weierstrass Approximation Thm, we see that

(3) also holds for any continuous functions on $[0,1]$

We still need to show that (3) also holds for $f \in \mathcal{R}[0,1)$.

Let $f \in \mathcal{R}[0,1]$. Let $\varepsilon > 0$. Then we can find

continuous functions h^+, h_- such that

$$h_- \le f \le h^+$$

and

$$\int h^+ - f\,dx < \varepsilon, \quad \int f - h_-\,dx < \varepsilon.$$

Then

$$\overline{\lim} \frac{1}{N} \sum_{n=1}^{N} f(x_n) \le \overline{\lim} \frac{1}{N} \sum_{n=1}^{N} h^+(x_n)$$

$$= \int_0^1 h^+\,dx < \int_0^1 f\,dx + \varepsilon$$

Similarly, $\underline{\lim} \frac{1}{N} \sum_{n=1}^{N} f(x_n) > \int_0^1 f\,dx - \varepsilon.$

Hence (3) holds for $f \in \mathcal{R}[0,1)$. In particular, (3) holds for $f = \chi_{(a,b)}$, $(a,b) \subset [0,1)$. That is, (x_n) is equidistributed. $\blacksquare$

§ 4.4 A continuous but nowhere differentiable functions.

- Riemann (1861) constructed the following function

$$R(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n^2 x), \quad x \in \mathbb{R}.$$

It is clear that R is cts on $\mathbb{R}$. Riemann guessed that $R(x)$ is nowhere diff.

- Hardy (1916): $R(x)$ is not diff at x if $\frac{x}{\pi}$ is irrational.

  Gerver (1969): $R(x)$ is diff $\Leftrightarrow$ $\frac{x}{\pi} = \frac{p}{q}$, p, q odd

- Weierstrass gave the following first example of cts but nowhere diff functions:

$$W(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x), \quad \text{where } \begin{array}{l} a>1, 0<b<1, \\ ab > 1+ 3\pi/2 \end{array}$$

Remark: People know nowadays that "$ab > 1 + \frac{3\pi}{2}$" can be weaken to $ab > 1$.

Here we prove a special version of Weierstrass's result.

Thm 3. Let $0<\alpha<1$. Then

$$f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i 2^n x}, \quad x \in \mathbb{R}$$

is cts but nowhere differentiable.

Idea: Let $g \in \mathcal{R}[-\pi,\pi]$, define for $N \in \mathbb{N}$,

(delayed means) $\Delta_N(g)(x) = 2\sigma_{2N}(g)(x) - \sigma_N(g)(x)$, where

$$\sigma_N(g)(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{g}(n) e^{inx}.$$

By a direct calculation,

$$\Delta_N(g)(x) = 2\sigma_{2N}(g)(x) - \sigma_N(g)(x)$$

$$= 2 \cdot \sum_{|n| \le 2N} \left(1 - \frac{|n|}{2N}\right) \hat{g}(n) e^{inx}$$

$$- \sum_{|n| \le N} \left(1 - \frac{|n|}{N}\right) \hat{g}(n) e^{inx}$$

$$= \sum_{|n| \le N} \hat{g}(n) e^{inx} + \sum_{N < |n| \le 2N} 2\left(1 - \frac{|n|}{2N}\right) \hat{g}(n) e^{inx}$$

$$= S_N(g)(x) + \sum_{N < |n| < 2N} 2\left(1 - \frac{|n|}{2N}\right) \hat{g}(n) e^{inx}.$$

Fact: ① If $N = 2^m$, then

$$\Delta_N(f_\alpha) = S_N(f_\alpha) + \sum_{2^m < |n| < 2^{m+1}} 2\left(1 - \frac{|n|}{2^{m+1}}\right) \hat{f_\alpha}(n) e^{inx}$$

$$= S_N(f_\alpha)$$

$$= \sum_{n=0}^{m} 2^{-n\alpha} \cdot e^{i 2^n x}$$

② In particular, for $N = 2^m$,

$$\Delta_{2N}(f_\alpha)(x) - \Delta_N(f_\alpha)(x) = 2^{-(m+1)\alpha} e^{i 2^{m+1} x}.$$

Prop 4. For any $g \in \mathcal{R}[-\pi,\pi]$, if g is diff at x_0,

then

$$\Delta_N(g)'(x_0) = O(\log N).$$

$\hookrightarrow$ Landau's big O notation.

(it means that $\exists$ a const C s.t

$|LHS| \leq C \cdot \log N$ for $N > 1$)

Now we prove that Prop 4 implies Thm 3.

Proof of Thm 3:

Suppose on the contrary that f_α is diff at x_0.

Then by Prop 4,

$$\Delta_N(f_\alpha)'(x_0) = O(\log N).$$

Hence

$$\Delta_{2N}(f_\alpha)'(x_0) - \Delta_N(f_\alpha)'(x_0) = O(\log N + \log 2)$$

Taking $N = 2^m$ gives

$$\frac{d}{dx}\left(2^{-(m+1)\alpha} \cdot e^{i 2^{m+1} x}\right)\Big|_{x_0} = O(m \log 2 + \log 2).$$

but it means

$$i\, 2^{(m+1)(1-\alpha)} \cdot e^{i 2^{m+1} x_0} = O\left((m+1)\log 2\right)$$

that is,

$$2^{(m+1)(1-\alpha)} = O\left((m+1)\log 2\right),$$

leading to a contradiction.

$\blacksquare$

In the end we prove Prop 4.

<u>Lemma 5</u>: Let $F_N(x) = \sum_{|n|\le N} \left(1-\frac{|n|}{N}\right) e^{inx}$

$$= \frac{\sin^2 \frac{Nx}{2}}{N \sin^2 \frac{x}{2}}.$$

Then $\exists$ a constant $A>0$ such that

$$|F_N'(x)| \le AN^2, \quad |F_N'(x)| \le \frac{A}{x^2}$$

for any $x \in [-\pi, \pi]$

pf. $\quad F_N'(x) = \sum_{|n|\le N} in \cdot \left(1-\frac{|n|}{N}\right) e^{inx}$

Then $\left| F_N'(x) \right| \le (2N+1)N < AN^2$.

Now

$$F_N'(x) = \frac{\sin\left(\frac{N}{2}x\right)\cos\left(\frac{N}{2}x\right)}{\sin^2\frac{x}{2}} - \frac{\sin^2\left(\frac{N}{2}x\right)\cos\left(\frac{x}{2}\right)}{N\sin^3\left(\frac{x}{2}\right)}$$

Hence

$$\left| F_N'(x) \right| \le \frac{1}{\sin^2\frac{x}{2}} + \frac{\left|\frac{N}{2}x\right|}{N\left|\sin^3\left(\frac{x}{2}\right)\right|},$$ where we use $|\sin x| \le |x|$.

$$\le \frac{A}{x^2}$$ by using

$|\sin x| \geqslant \text{const}\cdot|x|$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\blacksquare$